Problem

A materials laboratory studies a device that balances internal energy states. Each state of the device is described by four real parameters (a, b, c, d). When two states are applied in succession, the resulting state is obtained according to the rule

$$(a, b, c, d) \star (a', b', c', d') \;\; = (aa' + bc',\; ab' + bd',\; ca' + dc',\; cb' + dd'). \qquad (\star)$$

To each state the laboratory assigns a non-negative quantity $E(a, b, c, d)$, called its *energy level*, satisfying the composition law

$$E(X \star Y) = E(X)\, E(Y) \qquad \text{for all states } X, Y. \qquad (\dagger)$$

Experimental measurements reveal the following facts:

(i) $E(a, 0, 0, d) = ad, \qquad$ for all $a, d > 0$.

(ii) $E(1, b, -b, 1) = 1 \qquad$ for all $b \in \mathbb{R}$.

Determine the energy function $E(a, b, c, d)$ explicitly and describe the largest collection of states for which it is defined.