

An Iterated symplectic map for understanding Hamiltonian Chaos

1 Introduction

Chaos has intrigued, mesmerized and challenged generations of physicists and mathematicians. Poincar's work gave a glimpse of how complex and beautiful dynamical structures are, in chaotic systems. Robert May observed the sequence of bifurcations in a simple one dimensional chaotic map which was later analyzed in the powerful theory of universality by Feigenbaum [1]. Since then the Feigenbaum constant has been observed in many physical and mathematical models [2] [3]. There also have been new approaches in understanding the periodic orbit structures, where periodic orbits or Hamiltonian systems are studied in terms of equivalent orbits in dissipative dynamics [4]. However there are still multitude of unanswered questions pertaining Chaos. Here I list a few of them.

- (1) Do all Hamiltonian systems with standard kinetic energy term and polynomial potentials always have stable tori in some part of the phase space irrespective of nonlinearity strength?
- (2) What is the nature of universality in Hamiltonian dynamics [3] ?
- (3) Can one identify the location of stable tori on an energy surface using purely analytical tools ?
- (4) Are there any general patterns in appearance of periodic orbits as the nonlinearity parameter is varied, like those given by the Sarkovskii's theorem [5]?

With the aim of getting more insights into the manner in which the orbit structure of a chaotic system changes with increase in the non-linearity parameter, I propose to study the symplectic maps derived by considering Poincar sections on the symmetry lines for the Henon-Heiles system. The aim is to study the geometry of the iterated maps so far as possible with a geometric view similar to the logistic map, to get insights into some of the above mentioned questions.

2 The map

I work with the Hamiltonian

$$H = \frac{p_1^2 + P_2^2}{2} + \epsilon \left(x^2 y - \frac{y^3}{3} \right) \quad (1)$$

This Hamiltonian has potential with 3 fold symmetry. The Poincar map of this Hamiltonian has been studied using various numerical and analytical tools over a long time, some of the studies can be found in the collection of reprints edited by Hao Bai Lin [6]. The map I propose to work with is similar but with a different geometric representation.

Consider the symmetry line $x = 0$, let us call this line S_1 . On a given energy surface if the value of the momenta are specified on S_1 then that fixes an orbit (with a multiplicity of 3, since y is an implicit cubic function of energy E and momenta p_1 and p_2 , however while studying the map numerically one can fix the branch of y value one works with). An orbit started from a point of the symmetry line with the given momenta will intersect the nearest symmetry line at the angle $\frac{2\pi}{3}$. The intersection point can be mapped back to an equivalent point on S_1 using an appropriate rotation. Thus if we denote the momenta on the initial point $(p_1(0), p_2(0))$ and the momenta of the new point mapped on S_1 as $(p_1(1), p_2(1))$, then one would arrive at a map of the following form,

$$\begin{aligned} p_1(1) &= f(p_1(0), p_2(0)) \\ p_2(1) &= g(p_1(0), p_2(0)) \end{aligned}$$

The map has a nontrivial structure as it is not defined by a function but rather by a relation, since starting with one initial point and using different choices of the y coordinate value one may find 3 different mapped values. Further every point (p_1, p_2) has a closed curve in the phase space as it's pre-image. These intricacies of the map in themselves make it an interesting map to study.

3 The Iterations

Now let us consider the map. The second iteration is be of the form

$$\begin{aligned} p_1(n+2) &= f(f(p_1(n), p_2(n)), g(p_1(n), p_2(n))) \\ p_2(n+2) &= g(f(p_1(n), p_2(n)), g(p_1(n), p_2(n))) \end{aligned}$$

Note that each relation f and g describes a folded surface in the 3D space. The condition for fixed point are,

$$\begin{aligned} p_1^* &= f(p_1^*, p_2^*) \\ p_2^* &= g(p_1^*, p_2^*) \end{aligned}$$

The fixed point will correspond to one of the simplest periodic orbit, the circular orbit. The higher iterations of the map would lead to more complex periodic orbits. The change in the shape of the surfaces described by f and g would lead to bifurcations.

4 The work plan

In the proposed 3 year project I wish to carry out the following steps.

- (1) Step 1:(6 months): construct the maps f and g numerically on an energy surface and find approximate analytical expressions using curve fitting. Use linear and nonlinear regression methods for the purpose.
- (2) Step 2:(6 months): Look for symbolic dynamics rules that decide how iterations from one branch of the surface to another are carried out. Look for feasibility of pictorial iteration schemes like the logistic map.
- (3) Step 3:(1 year): Study the change in the shape of the surfaces defined by f and g numerically as ϵ (or energy) is changed and attempt to model that change. Make use of regression techniques. If a good model can be identified study bifurcation cascade in the model.
- (4) Step 4:(1 year): Continue previous study. Study various geometric representations of iterations of the map, along projections as well as in 3D.

References

- [1] Feigenbaum, Mitchell J. "Universal Behavior in Nonlinear Systems". *Physica*. 7D: 1639 (1983)
- [2] Cvitanovic P., "Universality in Chaos", *Acta Phys. Polonica* A65, 203 (1984)
- [3] Lakshminarayan A. et. al. "Local scaling in homogeneous Hamiltonian systems", *Phys. Rev. Lett.* 76.396 (1996)
- [4] Magnitskii N. A., "Non classical approach to the analysis of Hamiltonian and conservative systems", *Computational Math. and Modeling*, 24 (2) 221-251 (2013)
- [5] Hurwitz J. "Sarkovskii's theorem and it's converse", expository article at, <https://ocw.mit.edu/courses/mathematics/18-091-mathematical-exposition.../s4.pdf> (2005)
- [6] Hao Bai Lin *Chaos, an introduction and reprints volume*, *World Scientific Publishing Co., Singapore*, 1 1-576 (1984)